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Algebra 1
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# Midterm

**No credit will be given to unjustified answers. Justify all your answers completely.** (Either with a proof or with a counter example) unless mentioned differently. No step should be a mystery or bring a question. The grader cannot be expected to work his way through a sprawling mess of identities presented without a coherent narrative through line. If he can't make sense of it in finite time you could loose serious points. Coherent, readable exposition of your work is half the job in mathematics. You will loose serious points if your exposition is messy, incomplete, uses mathematical symbols not adapted...

**Problem 1 :**
Let $\mathbb{C}^\times$ be the multiplicative group of nonzero complex numbers. Let $\mathbb{P}$ be the set of positive real numbers. Let $\mathbb{S}$ be the multiplicative group of all complex numbers with absolute value 1.

1. Prove that $\mathbb{P}$ is a subgroup of $\mathbb{C}^\times$.
2. Prove that $\mathbb{S}$ is a subgroup of $\mathbb{C}^\times$.
3. Let
$$\begin{array}{cccc} \tilde{c}: & \mathbb{C}^\times/\mathbb{P} & \to & \mathbb{S} \\ & z\mathbb{P} & \mapsto & \frac{z}{|z|} \end{array}$$
where $|z|$ is the module of $z \in \mathbb{C}$.
   (a) Prove that $\tilde{c}$ is well defined. That is, if $z\mathbb{P} = z'\mathbb{P}$. Prove that $\frac{z}{|z|} = \frac{z'}{|z'|}$.
   (b) Prove that $\tilde{c}$ is a homomorphism.
   (c) Compute the kernel of $\tilde{c}$. Is $\tilde{c}$ one-to-one ?
   (d) Compute the range of $\tilde{c}$. Is $\tilde{c}$ onto $\mathbb{S}$ ?
   (e) Is it an isomorphism ?

**Solution :**

1. $\mathbb{P} \subseteq \mathbb{C}^\times$ and $p = p + i0 \in \mathbb{C}^\times$ for any $p \in \mathbb{P}$. The identity of $\mathbb{C}^\times$ 1 is a positive number. The product of two positive number being positive $\mathbb{P}$ is closed under multiplication. The inverse of a positive number $p$ is $1/p$ a positive number.
2. $\mathbb{S} \subseteq \mathbb{C}^\times$ and $s \in \mathbb{C}^\times$ for any $s \in \mathbb{S}$ as $|0| = 0$. The identity of $\mathbb{C}^\times$ 1 is $\mathbb{S}$ as $|1| = 1$. Let $s_1, s_2 \in \mathbb{S}$, that is $|s_1| = 1$ and $|s_2| = 1$ then $|s_1 s_2| = |s_1||s_2| = 1.1 = 1$. Thus, $\mathbb{S}$ is closed under multiplication. The inverse of $s \in \mathbb{S}$ is $1/s$ is in $\mathbb{S}$ indeed as $|s| = 1$ then $|1/s| = 1/|s| = 1$.

3. Let

$$\begin{array}{rccc} \tilde{c}: & \mathbb{C}^\times/\mathbb{P} & \to & \mathbb{S} \\ & z\mathbb{P} & \mapsto & \frac{z}{|z|} \end{array}$$

where $|z|$ is the module of $z \in \mathbb{C}$.

(a) Suppose that $z\mathbb{P} = z'\mathbb{P}$. That is there is a $p \in mathbbP$ such that $z = z'p$. Then

$$\frac{z}{|z|} = \frac{z'p}{|zp|} = \frac{z'p}{|z'|p} = \frac{z'}{|z'|}.$$

Thus $\tilde{c}$ is well defined.

(b) Let $z\mathbb{P}, z'\mathbb{P} \in \mathbb{C}^\times/\mathbb{P}$,

$$\begin{array}{rcc} \tilde{c}(z\mathbb{P} \cdot z'\mathbb{P}) & = & \tilde{c}(z \cdot z'\mathbb{P}) \\ & = & \frac{zz'}{|zz'|} = \frac{z}{|z|}\frac{z'}{|z'|} \\ & = & \tilde{c}(z\mathbb{P}) \cdot \tilde{c}(z'\mathbb{P}) \end{array}$$

Then, $\tilde{c}$ is a homomorphism.

(c) The kernel of $\tilde{c}$ is :

$$\begin{array}{rcc} ker(\tilde{c}) & = & \{z\mathbb{P} \in \mathbb{C}^\times/\mathbb{P} | \tilde{c}(z\mathbb{P}) = 1\} \\ & = & \{z\mathbb{P} \in \mathbb{C}^\times/\mathbb{P} | |z| = z\} \\ & = & \{z\mathbb{P} \in \mathbb{C}^\times/\mathbb{P} | z \in \mathbb{P}\} \\ & = & \{e\mathbb{P}\} \end{array}$$

As $\tilde{c}$ is an homomorphism then it is one-to-one.

(d) The range of $\tilde{c}$ is

$$\begin{array}{rcc} range(\tilde{c}) & = & \{\tilde{c}(z\mathbb{P}) | z\mathbb{P} \in \mathbb{C}^\times/\mathbb{P}\} \\ & = & \{|z|/z : z\mathbb{P} \in \mathbb{C}^\times/\mathbb{P}\} \end{array}$$

Note that $|z/|z|| = |z|/|z| = 1$ this $z/|z| \in \mathbb{S}$.
For any $s \in \mathbb{S}$ then $s/|s| = s/1 = s = \tilde{c}(s\mathbb{P})$. Thus, $range(\tilde{c}) = \mathbb{S}$. As a consequence, $\tilde{c}$ is onto $\mathbb{S}$.

(e) Is it an isomorphism as it is an bijective homomorphism.

**Problem 2 :**

1. Give an exhausted list of the element of $S_3$.
2. Is $S_3$ abelian ? Justify.
3. Is $S_3$ cyclic ? Justify.
4. Give all the elements of order $2$ in $S_3$. What about elements of order $4$ ?
5. Let $H = < (1, 2) >$

(a) Describe $H$. How many elements are there in $H$ ? To which well known group is $H$ isomorphic to ? (Hint : Think about the classification of cyclic group.)

(b) Describe the left and the coset of the permutation $(1,2,3)$ for $H$. Is $H$ a normal subgroup of $S_3$ ?

6. Let $H' = \{e, (1,3,2), (1,2,3)\}$.

   (a) Doing the table of $H'$, prove that $H'$ is a subgroup of $S_3$.

   (b) Is $H'$ cyclic ? If yes, give a generator of $H'$. To which well known group is $H'$ isomorphic to ? (Hint : Think about the classification of cyclic group.)

   (c) Describe the element of the quotient space $S_3/H'$. How many distinct elements are there in $S_3/H'$ ?

   (d) Let $\sigma \in H'$, we define

$$\begin{array}{rccc} \phi_\sigma : & H' & \to & H' \\ & f & \mapsto & \sigma \circ f \end{array}$$

   Show that $\phi_\sigma$ is a bijection by finding its inverse. Is $\phi_\sigma$ a homomorphism ?

   (e) Denote $Per(H')$ be the set of all the bijection of $H'$. It is a group with the composition operation. Let

$$\begin{array}{rccc} \Phi : & H' & \to & Per(H') \\ & \sigma & \mapsto & \phi_\sigma \end{array}$$

   i. Prove that $\Phi$ is an homomorphism.

   ii. Compute the kernel. Is $\Phi$ one-to-one ?

   iii. Compute the range. Is $\Phi$ onto $Per(H')$ ?

   iv. Is $\Phi$ an isomorphism ?

**Solution :**

1. $S_3 = \{id, (1,2), (2,3), (1,3), (1,2,3), (1,3,2)\}$ (Note that $|S_3| = 3 \cdot 2 = 6$.)
2. Note that $(1,2)(1,3) = (1,3,2)$ and $(1,3)(1,2) = (1,2,3)$ thus $(1,2)(1,3) \neq (1,3)(1,2)$ And, $S_3$ is not abelian.
3. $S_3$ is cyclic if there is $\sigma \in S_3$ such that $S_3 =< \sigma >$. In particular, we need an element $\sigma \in S_3$, with order $6 = |S_3|$.
   Note that $o((1,2)) = o((2,3)) = o((1,3)) = 2$ indeed $(1,2)^2 = (1,3)^3 = (2,3)^2 = Id$.
   and $o((1,2,3)) = o((1,3,2)) = 3$ indeed $(1,2,3)^2 = (1,3,2)$ and $(1,2,3)^3 = Id$ and $(1,3,2)^2 = (1,2,3)$ and $(1,3,2)^3 = Id$ Thus $S_3$ os not cyclic
4. As seen in the previous question, the element of order $2$ are $(1,2)$, $(1,3)$ and $(2,3)$ and there is no elements of order $4$.
5. Let $H =< (1,2) >$

   (a) $H = \{(1,2)^k, k = \mathbb{Z}\}$ as $(1,2)$ has order $2$, we have $H = \{Id, (1,2)\}$ and

$$H \simeq \mathbb{Z}/2\mathbb{Z}$$

   (b) The left coset is $(1,2,3)H = \{(1,2,3), (1,2,3)(1,2)\} = \{(1,2,3), (1,3)\}$ and the right coset is $H(1,2,3) = \{(1,2,3), (1,2)(1,2,3)\} = \{(1,2,3), (2,3)\}$.Thus Since $(1,2,3)H \neq H(1,2,3)$, then $H$ is not normal in $G$ and as a consequence, $G/H$ is not a group.

6. Let $H' = \{e, (1,3,2), (1,2,3)\}$.

(a) The table of $H'$ is given by

| | $e$ | $(1,2,3)$ | $(1,3,2)$ |
|---|---|---|---|
| $e$ | $e$ | $(1,2,3)$ | $(1,3,2)$ |
| $(1,2,3)$ | $(1,2,3)$ | $(1,3,2)$ | $e$ |
| $(1,3,2)$ | $(1,3,2)$ | $e$ | $(1,2,3)$ |

As we can see from the table, $e \in H'$, $(1,2,3)^{-1} = (1,3,2)$ and $(1,3,2)^{-1} = (1,2,3)$, and the product of two element in $H'$ is an element of $H'$. Thus, $H'$ is a group.

(b) Note that $(1,3,2)^2 = (1,2,3)$ and $(1,3,2)^3 = Id$ thus $(1,3,2)$ is a generator of $H'$ and $H'$ is cyclic. Finally, $H' \simeq \mathbb{Z}/3\mathbb{Z}$.

(c)
$$S_3/H' = \{\sigma H' | \sigma \in S_3\}$$

For any $\sigma \in H'$, $\sigma H' = H'$ and

$$(1,2)H' = (1,3)H' = (2,3)H' = \{(1,2), (1,3), (2,3)\}$$

Thus
$$S_3/H' = \{eH', \{(1,2), (1,3), (2,3)\}\}$$

and $|S_3/H'| = 2$.

(d) Let $\sigma \in H'$, we define

$$\begin{array}{cccc} \phi_\sigma : & H' & \to & H' \\ & f & \mapsto & \sigma \circ f \end{array}$$

Define

$$\begin{array}{cccc} \phi_\sigma^{-1} : & H' & \to & H' \\ & f & \mapsto & \sigma^{-1} \circ f \end{array}$$

$$\phi_\sigma \circ \phi_{\sigma^{-1}} = \phi_{\sigma^{-1}} \circ \phi_\sigma = Id$$

Indeed, for $f \in H'$,

$$\phi_\sigma \circ \phi_{\sigma^{-1}}(f) = \sigma \circ \sigma^{-1} \circ f = Id \circ f = f$$

$$\phi_\sigma^{-1} \circ \phi_\sigma(f) = \sigma^{-1} \circ \sigma \circ f = Id \circ f = f$$

Thus $\phi_\sigma$ is a bijection since it has an inverse. If $\sigma = Id$ then $\phi_e = Id$ is an homomorphism.
If $\sigma = (1,2,3)$ then

$$\phi_\sigma((1,2)\circ(2,3)) = \phi_\sigma((1,2)\circ(2,3)) = \phi_\sigma((2,3,1)) = (1,2,3)\circ(2,3,1) = (1,3,2)$$

$$\phi_\sigma((1,2))\circ\phi_\sigma((2,3)) = (1,2,3)\circ(1,2)\circ(1,2,3)\circ(2,3) = (1,3)\circ(1,2) = (1,2,3)$$

Thus if $\sigma = (1,2,3)$ then

$$\phi_\sigma((1,2)\circ(2,3)) = \phi_\sigma((1,2)\circ(2,3)) = \phi_\sigma((2,3,1)) = (1,2,3)\circ(2,3,1) = (1,3,2)$$

$$\phi_\sigma((1,2))\circ\phi_\sigma((2,3)) = (1,2,3)\circ(1,2)\circ(1,2,3)\circ(2,3) = (1,3)\circ(1,2) = (1,2,3)$$

$$\phi_\sigma((1,2)\circ(2,3)) \neq \phi_\sigma((1,2))\circ\phi_\sigma((2,3))$$

and $\phi_\sigma$ is not a homomorphism. Similarly, if $\sigma = (1,3,2)$ then

$$\phi_\sigma((1,2)\circ(2,3)) = \phi_\sigma((1,2)\circ(2,3)) = \phi_\sigma((2,3,1)) = (1,3,2)\circ(2,3,1) = Id$$

$$\phi_\sigma((1,2))\circ\phi_\sigma((2,3)) = (1,3,2)\circ(1,2)\circ(1,3,2)\circ(2,3) = (2,3)\circ(1,3) = (1,2,3)$$

$$\phi_\sigma((1,2)\circ(2,3)) \neq \phi_\sigma((1,2))\circ\phi_\sigma((2,3))$$

and $\phi_\sigma$ is not a homomorphism.

(e) Denote $Per(H')$ be the set of all the bijection of $H'$. It is a group with the composition operation. Let

$$\begin{array}{rccc} \Phi: & H' & \rightarrow & Per(H') \\ & \sigma & \mapsto & \phi_\sigma \end{array}$$

i. Let $\sigma$ and$\sigma' \in H'$, Then

$$\Phi(\sigma\circ\sigma') = \Phi(\sigma)\circ\Phi(\sigma')$$

Indeed, let $f \in H'$,

$$\Phi(\sigma\cdot\sigma')(f) = \sigma\cdot\sigma'\circ f = \Phi(\sigma)(\sigma'\circ f) = \Phi(\sigma)\circ\Phi(\sigma')(f)$$

Thus, $\Phi$ is an homomorphism.

ii.

$$\begin{array}{rcc} ker(\Phi) & = & \{\sigma\in H'|\phi(\sigma) = Id\} \\ & = & \{\sigma\in H'|\sigma\circ h = h, \forall h\in H'\} \\ & = & \{\sigma\in H'|\sigma = Id, \forall h\in H'\} \\ & = & \{Id\} \end{array}$$

As $\sigma\circ h = h$ implies $\sigma = Id$ by multiplying the equality by $h^{-1}$ on the right. Thus as $\Phi$ is a homomorphism, $\Phi$ one-to-one.

iii.

$$\begin{array}{rcc} range(\Phi) & = & \{\phi(\sigma)|\sigma\in H'\} \\ & = & \{\Phi_I d, \Phi_{(1,2,3)}, \Phi_{(1,3,2)}\} \end{array}$$

Note that

$$\Phi_I d = Id$$

$$\Phi_{(1,2,3)}(Id) = (1,2,3), \Phi_{(1,2,3)}((1,2,3)) = (1,3,2) \text{ and } \Phi_{(1,2,3)}((1,3,2)) = Id$$

$$\Phi_{(1,3,2)}(Id) = (1,3,2), \Phi_{(1,3,2)}((1,2,3)) = Id \text{ and } \Phi_{(1,3,2)}((1,3,2)) = (1,2,3)$$

$$|Per(H')| = 3! = 6 \neq |range(\Phi)|$$

Thus $\Phi$ is not onto $Per(H')$.

iv. $\Phi$ is not an isomorphism as $\Phi$ is not onto.